GROUP THEORY

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1.**Introduction**: Group Theory is the study of Algebraic structures.

2.**Algebraic structure**: A non-empty set G equipped with one or more binary Operations is called an algebraic structure.

If O is the binary operation on G, then the algebraic structure is written as (G,0).

(N,+),(Q,-),(R,+) are algebraic structures

3. **Binary operation**: A Binary operation on a set is a calculation that combines two elements of the set to produce another element of the set.

Let R be the set of real numbers and addition(+), subtraction(-), union(U), Intersection are binary operations in R.

Group Definition:

A group is a set G, together with an operation • that combines any two elements a and b to form another element denoted as $a \bullet b$ or ab.

A set is said to be a group with operation $(G, \bullet)$, it must satisfy four requirements known as the *group axioms:*

1.Closure law:

For all a, b in G, the result of the operation, $a \bullet b$ is also in G.

2. Associative law:

For all a, b and c in G, $(a \bullet b) \bullet c = a \bullet (b \bullet c)$.

3. Identity element:

There exists an element e in G such that, for every element a in G, the equation $e \bullet a = a \bullet e = a$ holds.

4. Inverse element:

For each a in G, there exists an element b in G, commonly denoted a^{-1} , such that $a \bullet b = b \bullet a = e$, where e is the identity element.

Example:

To show that set of all positive rational numbers form an abelian group under the Composition defined by O such that aob=ab/2

Proof:

To show that (Q+,0) is a group it must satisfy the following properties

1.CLosure Property:

Since for every a,b ϵQ+,(ab)/2 is also in Q+
therefore Q+ is closed with respect to operation o.

2.Assosiative Property:

Let a,b,c ϵ Q+.Then
(a*b)*c=(ab/2)*c=[(ab)/2]c/2=a([(bc/2)]/2=a*(bc/2)=a*(b*c)

3.Existense of identity: The number e will be identity element
if eϵQ+ and if e*a=a=a*e VaϵQ+
Now e*a=a
which implies (ea)/2 =a
e=2

4.Existense of inverse:

Let a be any element of Q+ If the number b is to be the inverse of a , then we must have
b*a=e=2
(ba)/2=2
b=4/a Hence(Q+,o) is an abelian group.

Subgroups:

Let (G,.) be a group. Let H be a non empty subset of G such that (H,.) be a group. Then H is called a subgroup of G.

Theorem:

If H1 and H2 are two subgroups then the intersection of two subgroups $H1 \cap H2$ also a subgroup of G

Proof:-

You should know the subgroup criterion: A subset H of a group is a subgroup iff for a,b∈H implies $ab^{-1} \in H$. Hence we have just shown that H is a subgroup of G.

Let G be a group, $H1, H2$ subgroups of G

Let e be the identity in G

$e \in H1$ and $e \in H2$, implies $e \in H1 \cap H2$. i.e $H1 \cap H2 \neq \emptyset$.

Let $a \in H1 \cap H2$, $b \in H1 \cap H2$

$a \in H1$ and $a \in H2$ and $b \in H1$, $b \in H2$

Since H1 is a subgroup, for $a \in H1$ and $b \in H1$ implies $ab^{-1} \in H1$.

Similarly, for $a \in H2$ and $b \in H2$ implies $ab^{-1} \in H2$

hence $ab^{-1} \in H1 \cap H2$

Thus we have $a \in H1 \cap H2$, $b \in H1 \cap H2$ implies $ab^{-1} \in H1 \cap H2$

Hence $H1 \cap H2$ is a subgroup of G.

Cosets and Lagrange's theorem:

Let G be a group and H be a subgroup of G Let a ∈G then the set aH={ah/h ∈H} is called a left coset of H in G generated by a and the set Ha={ha/h ∈H} is called a right coset of H in G generated by a.

Lagrange's theorem:

The order of a subgroup of a finite group divides the order of the group

Proof:-

Since H is a subgroup of a finite group G, H is finite

1) If H=G then O(H)/O(G)
2) If H#G. Let O(G)=n and O(H)=m

   we know that every right coset of H in G has the same number of elements and the number of right cosets of H in G is finite.

   If Ha,Hb,Hc…………are right cosets of H in G then

   O(Ha)=O(Hb)=O(Hc)=…………………O(H)=m

   Let the number of distinct right cosets of H of G be k

   All these right cosets are disjoint and induce a partition of G

   Therefore, O(G)=O(Ha)+O(Hb)+O(Hc)=…………………..O(H) (k terms)

   = m+m+m+…………………………………..(k times)

   Implies n=km implies k=n/m

Therefore, O(H) divides O(G) i.e O(H)/O(G).

Normal Subgroups:

A subgroup H of a group G is said to be a normal subgroup of G if for $x \in G$ *and* $h \in H$, $xhx-1 \in H$

Theorem :

A subgroup H of a group *G* is a normal subgroup of *G* if and only if each left coset of *N* in *G* is a right coset of *N* in *G*

Proof: Let *H* be a normal subgroup of *G*

Then $xHx-1=H \ \forall x \in G$

$\Rightarrow (xHx-1)x=Hx \ \forall x \in G$

$\Rightarrow xH=Hx \ \forall x \in G$

$\Rightarrow\Rightarrow$ each left coset *of H in G* is a right coset *of H in G*

Conversely, let each left coset of *H* in G be a right coset of N in G. This means that if *x* is any element of G, then the left coset is also a right coset.

Now e∈H, and therefore xe=x∈xH.

So *x* must also belong to that right coset which is equal to left coset *xH*. But *x* is an element of the right coset Hx, and two right cosets are either disjointe or identical, i.e. if two right cosets contain one common element then they are identical. Therefore Hx is the unique right coset which is equal to the left coset xH.

Therefore, we have

$xH=Hx\forall x\in$

$\Rightarrow xHx-1=Hxx-1\forall x\in G \Rightarrow xHx-1=Hxx-1\forall x\in G$

$\Rightarrow xHx-1=H\forall x\in G$

$\Rightarrow\Rightarrow$ H is normal a subgroup of G

Homomorphisms: Let G, G' be two groups and f is a mapping from G into G'. If for a,b $\in$G, f(a.b)=f(a).f(b).

Fundamental Theorem on Homomorphisms of Groups:

If f:G $\rightarrow$G' is a homomorphism and onto with kernal K, then prove that G/K is isomorphic to G'

Proof:

Let f be a homomorphism from a group G onto group G'

Let kerf=k, then kerf is a normal subgroup of G and G/K is the quotient group of G by K

Now we have to prove that G/K is isomorphic toG'

For a $\in$G,ka $\in$G/k and f(a) $\in$G'

Define a mapping ϕ:G/k$\rightarrow$G' such that ϕ(Ka)=f(a) for a $\in$G

For a,b$\in$G, ka=kb implies $ab^{-1} \in K$ implies $f(ab^{-1})=e'$

$f(a)f(b^{-1})=e'$

$f(a)f(b^{-1})f(b)=e'\, f(b)$

$f(a)e'=f(b)$

$f(a)=f(b)$

$\phi(Ka)=\phi(Kb)$ ϕ is well defined

To prove ϕ is 1-1:

for a,b$\in$G Ka,Kb$\in$G/K. Now $\phi(Ka)=\phi(Kb)$

$f(a)=f(b)$

$f(a)f(b^{-1})=f(b)f(b^{-1})$

$f(ab^{-1})=e'$

$ab^{-1} \in K$

Ka=Kb thus ϕ is1-1

To prove ϕ is onto:

Since f:G $\rightarrow$G' is onto for a $\in$G such that f(a)=x

Ka $\in$G/K so ϕ (Ka)=f(a)=x hence ϕ is onto

To prove ϕ is a homomorphism:

for a,b $\in$G, Ka,Kb $\in$G/K

Now ϕ (Ka)(Kb)= ϕ (Kab)=f(ab)=f(a)f(b)= ϕ (Ka) ϕ (Kb)

Thus ϕ is an isomorphism from G/K onto G'

Permutation groups: A permutation is a one-one mapping of a non empty set onto itself

Cayley's theorem:

Every finite group G is isomorphic to a permutation group

Proof:

Let G be a finite group. Let a∈G then if for every x∈G, ax∈G

Now consider fa:G →G defined by fa(x)=ax for x ∈G

Therefore for x=y implies ax=ay implies fa(x)=fa(y)

Thus fa is well defined

fa is one-one since for x,y ∈G we have fa(x)=fa(y) →ax=ay →x=y

fa is onto since for x ∈G , $a^{-1}x$ ∈G such that fa($a^{-1}x$)=a($a^{-1}x$)=ex=x

Thus fa:G →G is one-one and onto

Thus fa is a permutation on G

Let G'={fa/a ∈G} i.e Let G' be the set of all permutations defined on G. We shall show that G' is a group with respect to permutation multiplication

Closure: for a,b ∈G, fa,fb ∈G'

fafb(x)=fa(fb(x))=fa(bx)=a(bx)=(ab)x=fab(x)

fafb=fab ∈G'

Associativity: for a,b,c ∈G and fa,fb,fc ∈G'

for x ∈G, (fafb)fc(x)=fa(fb)fc(x)=fafb(fc)(x)

=fafb(fc(x))=fa(fbfc(x))

thus (fafb)fc=fa(fbfc)

Existence of identity: Let e be the identity in G

fe ∈G and fefa=fea=fa'

Fafe=fae=fa'

Existence of inverse: if a ∈G then a^{-1}∈G

f a^{-1}∈G' and f a^{-1}fa=f a^{-1}a=fe'faf a^{-1}=fa a^{-1}=fe

Every element in G' is invertible

Thus G' is a group

Finally we show that G is isomorphic to G'

Consider ϕ :G →G' defined by ϕ (a)=fa for a ∈G

Φ is one-one since ϕ(a)= ϕ(b) implies fa=fb →fa(x)=fb(x) →ax=bx →a=b

Φ is onto since for fa ∈G,',a ∈G such that Φ(a)=fa

Φ is homomorphism since Φ (ab)=fab=fafb= Φ (a) Φ (b)

G is isomorphic to permutation group.

Conclusion:

1.Group theory plays a huge role in formulation of physics
2. In chemistry it is used to describe symmetries of crystal and molecular structures, quantum chemistry, spectroscopy
3. In Music
4. In Statistics
5. In computer science
6.In medical fields
7.In robotics
8. In graphics
9. In Biology


THANK YOU !!